

THE RATIONAL HOMOLOGY OF THE OUTER AUTOMORPHISM GROUP OF F_7

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ABSTRACT. We compute the homology groups $H_*(\text{Out}(F_7); \mathbb{Q})$ of the outer automorphism group of the free group of rank 7.

We produce in this manner the first rational homology classes of $\text{Out}(F_n)$ that are neither constant ($*$ = 0) nor Morita classes ($*$ = $2n - 4$).

1. INTRODUCTION

The homology groups $H_k(\text{Out}(F_n); \mathbb{Q})$ are intriguing objects. On the one hand, they are known to “stably vanish”, i.e. for all $n \in \mathbb{N}$ we have $H_k(\text{Out}(F_n); \mathbb{Q}) = 0$ as soon as k is large enough [3]. Hatcher and Vogtmann prove that the natural maps $H_k \text{Out}(F_n) \rightarrow H_k \text{Aut}(F_n)$ and $H_k \text{Aut}(F_n) \rightarrow H_k \text{Aut}(F_{n+1})$ are isomorphisms for $n \geq 2k + 2$ respectively $n \geq 2k + 4$, see [4, 5]. On the other hand, $H_k(\text{Out}(F_n); \mathbb{Q}) = 0$ for $k > 2n - 3$, since $\text{Out}(F_n)$ acts geometrically on a contractible space (the “spine of outer space”, see [2]) of dimension $2n - 3$. Combining these results, the only $k \geq 1$ for which $H_k(\text{Out}(F_n); \mathbb{Q})$ could possibly be non-zero are in the range $\frac{n}{2} - 2 < k \leq 2n - 3$. Morita conjectures in [9, page 390] that $H_{2n-3}(\text{Out}(F_n); \mathbb{Q})$ always vanishes; this would improve the upper bound to $k = 2n - 4$, and $H_{2n-4}(\text{Out}(F_n); \mathbb{Q})$ is also conjectured to be non-trivial.

We shall see that the first conjecture does not hold. Indeed, the first few values of $H_k(\text{Out}(F_n); \mathbb{Q})$ may be computed by a combination of human and computer work, and yield

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
2	1	0										
3	1	0	0	0								
4	1	0	0	0	1	0						
5	1	0	0	0	0	0	0	0				
6	1	0	0	0	0	0	0	0	1	0		
7	1	0	0	0	0	0	0	0	1	0	0	1

The values for $n \leq 6$ were computed by Ohashi in [12]. They reveal that, for $n \leq 6$, only the constant class ($k = 0$) and the Morita classes $k = 2n - 4$ yield non-trivial homology. The values for $n = 7$ are the object of this Note, and reveal that the picture changes radically:

Theorem. *The non-trivial homology groups $H_k(\text{Out}(F_7); \mathbb{Q})$ occur for $k \in \{0, 8, 11\}$ and are all 1-dimensional.*

Previously, only the rational Euler characteristic $\chi_{\mathbb{Q}}(\text{Out}(F_7)) = \sum (-1)^k \dim H_k(\text{Out}(F_7); \mathbb{Q})$ was known [10], and shown to be 1. These authors computed in fact the rational Euler characteristics up to $n = 11$ in that paper and the sequel [11].

2. METHODS

We make fundamental use of a construction of Kontsevich [6], explained in [1]. We follow the simplified description from [12].

Let F_n denote the free group of rank n . This parameter n is fixed once and for all, and will in fact be omitted from the notation as often as possible. An *admissible graph of rank n* is a

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graph G that is 2-connected (G remains connected even after an arbitrary edge is removed), without loops, with fundamental group isomorphic to F_n , and without vertex of valency ≤ 2 . Its *degree* is $\deg(G) := \sum_{v \in V(G)} (\deg(v) - 3)$. In particular, G has $2n - 2 - \deg(G)$ vertices and $3n - 3 - \deg(G)$ edges, and is trivalent if and only if $\deg(G) = 0$. If Φ is a collection of edges in a graph G , we denote by G/Φ the graph quotient, obtained by contracting all edges in Φ to points.

A *forested graph* is a pair (G, Φ) with Φ an oriented forest in G , namely an ordered collection of edges that do not form any cycle. We note that the symmetric group $\text{Sym}(k)$ acts on the set of forested graphs whose forest contains k edges, by permuting the forest's edges.

For $k \in \mathbb{N}$, let C_k denote the \mathbb{Q} -vector space spanned by isomorphism classes of forested graphs of rank n with a forest of size k , subject to the relation

$$(G, \pi\Phi) = (-1)^\pi (G, \Phi) \text{ for all } \pi \in \text{Sym}(k).$$

Note, in particular, that if $(G, \Phi) \sim (G, \pi\Phi)$ for an odd permutation π then $(G, \Phi) = 0$ in C_k . These spaces (C_*) form a chain complex for the differential $\partial = \partial_C - \partial_R$, defined respectively on $(G, \Phi) = (G, \{e_1, \dots, e_p\})$ by

$$\begin{aligned} \partial_C(G, \Phi) &= \sum_{i=1}^p (-1)^i (G/e_i, \Phi \setminus \{e_i\}), \\ \partial_R(G, \Phi) &= \sum_{i=1}^p (-1)^i (G, \Phi \setminus \{e_i\}), \end{aligned}$$

and the homology of (C_*, ∂) is $H_*(\text{Out}(F_n); \mathbb{Q})$.

The spaces C_k may be filtered by degree: let $F_p C_k$ denote the subspace spanned by forested graphs (G, Φ) with $\deg(G/\Phi) \leq p$. The differentials satisfy respectively

$$\partial_C(F_p C_k) \subseteq F_p C_{k-1}, \quad \partial_R(F_p C_k) \subseteq F_{p-1} C_{k-1}.$$

A spectral sequence argument gives

$$(1) \quad H_p(\text{Out}(F_n); \mathbb{Q}) = E_{p,0}^2 = \frac{\ker(\partial_C|_{F_p C_p}) \cap \ker(\partial_R|_{F_p C_p})}{\partial_R(\ker(\partial_C|_{F_{p+1} C_{p+1}}))}.$$

Note that if $(G, \Phi) \in F_p C_p$ then G is trivalent. We compute explicitly bases for the vector spaces $F_p C_p$, and matrices for the differentials ∂_C, ∂_R , to prove the theorem.

3. IMPLEMENTATION

We follow for $n = 7$ the procedure sketched in [12]. Using the software program **nauty** [8], we enumerate all trivalent graphs of rank n and vertex valencies ≥ 3 . The libraries in **nauty** produce a canonical ordering of a graph, and compute generators for its automorphism group. We then weed out the non-2-connected ones.

For given $p \in \mathbb{N}$, we then enumerate all p -element oriented forests in these graphs, and weed out those that admit an odd symmetry. These are stored as a basis for $F_p C_p$. Let a_p denote the dimension of $F_p C_p$.

For (G, Φ) a basis vector in $F_p C_p$, the forested graphs that appear as summands in $\partial_C(G, \Phi)$ and $\partial_R(G, \Phi)$ are numbered and stored in a hash table as they occur, and the matrices ∂_C and ∂_R are computed as sparse matrices with a_p columns.

The nullspace $\ker(\partial_C|_{F_p C_p})$ is then computed: let b_p denote its dimension; then the nullspace is stored as a sparse $(a_p \times b_p)$ -matrix N_p . The computation is greatly aided by the fact that ∂_C is a block matrix, whose row and column blocks are spanned by $\{(G, \Phi) : G/\Phi = G_0\}$ for all choices of the fully contracted graph G_0 . The matrices N_p are computed using the linear algebra library **linbox** [7], which provides exact linear algebra over \mathbb{Q} and finite fields.

Finally, the rank c_p of $\partial_R \circ N_p$ is computed, again using **linbox**. By (1), we have

$$\dim H_p(\text{Out}(F_n); \mathbb{Q}) = b_p - c_p - c_{p+1}.$$

For memory reasons (the computational requirements reached 200GB of RAM at its peak), some of these ranks were computed modulo a large prime (65521 and 65519 were used in two independent runs).

Computing modulo a prime can only reduce the rank; so that the values c_p we obtained are underestimates of the actual ranks of $\partial_R \circ N_p$. However, we also know *a priori* that $b_p - c_p - c_{p+1} \geq 0$ since it is the dimension of a vector space; and none of the c_p we computed can be increased without at the same time causing a homology dimension to become negative, so our reduction modulo a prime is legal.

For information, the parameters a_p, b_p, c_p for $n = 7$ are as follows:

p	0	1	2	3	4	5	6	7	8	9	10	11
a_p	365	3712	23227	$\approx 105k$	$\approx 348k$	$\approx 854k$	$\approx 1.6m$	$\approx 2.3m$	$\approx 2.6m$	$\approx 2.1m$	$\approx 1.2m$	$\approx 376k$
b_p	365	1784	5642	14766	28739	39033	38113	28588	16741	6931	1682	179
c_p	0	364	1420	4222	10544	18195	20838	17275	11313	5427	1504	178

The largest single matrix operations that had to be performed were computing the nullspace of a 2038511×536647 matrix (16 CPU hours) and the rank modulo 65519 of a (less sparse) 1355531×16741 matrix (10 CPU hours).

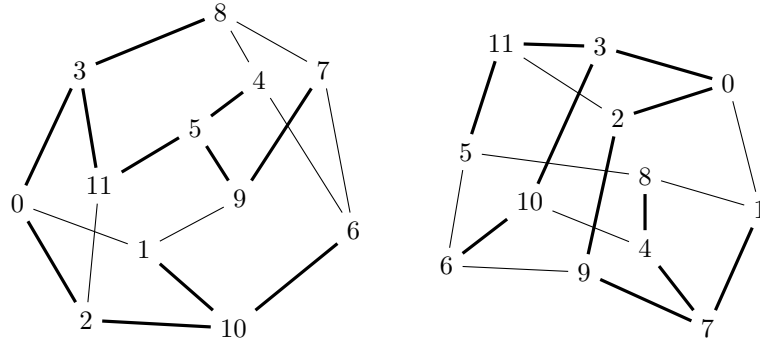
The source files used for the computations are available as supplemental material. Compilation requires `g++` version 4.7 or later, a functional `linbox` library, available from the site <http://www.linalg.org>, as well as the `nauty` program suite, available from the site <http://pallini.di.uniroma1.it>. It may also be directly downloaded and installed by typing `'make nauty25r9'` in the directory in which the sources were downloaded. Beware that the calculations required for $n = 7$ are prohibitive for most desktop computers.

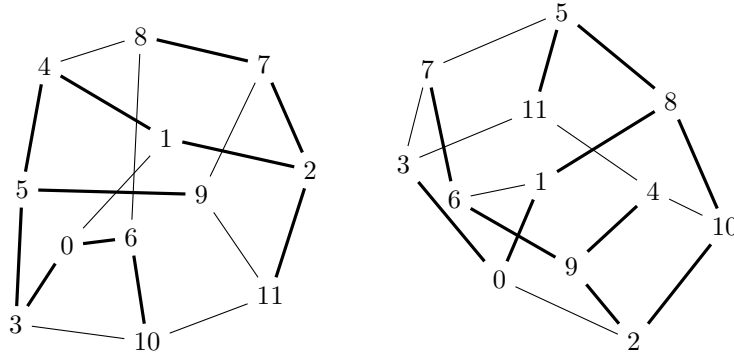
CONCLUSION

Computing the dimensions of the homology groups is only the first step in understanding them; much more interesting would be to know visually, or graph-theoretically, where these non-trivial classes come from.

It seems almost hopeless to describe, via computer experiments, the non-trivial class in degree 8. It may be possible, however, to arrive at a reasonable understanding of the non-trivial class in degree 11.

This class may be interpreted as a linear combination w of trivalent graphs on 12 vertices, each marked with an oriented spanning forest. There are 376365 such forested graphs that do not admit an odd symmetry. The class $w \in \mathbb{Q}^{376365}$ is an \mathbb{Z} -linear combination of 70398 different forested graphs, with coefficients in $\{\pm 1, \dots, \pm 16\}$. For example, eleven graphs occur with coefficient ± 13 ; four of them have indices 25273, 53069, 53239, 53610 respectively, and are, with the spanning tree in bold,





The coefficients of w , and corresponding graphs, are distributed as ancillary material in the file `w_cycle`, in format `'coefficient [edge1 edge2 ...]'`, where each edge is `'x-y'` or `'x+y'` to indicate whether the edge is absent or present in the forest. Edges always satisfy $x \leq y$, and the forest is oriented so that its edges are lexicographically ordered. Edges are numbered from 0 while graphs are numbered from 1. There are no multiple edges.

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